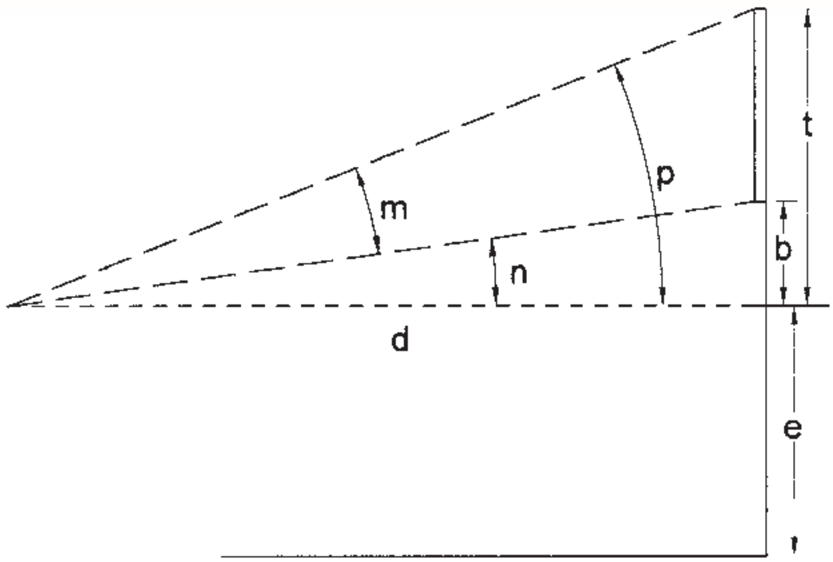
Solution to Problem 99



The solution is to maximize the angle m. Angle m = angle p – angle n.

$$\cot m = \cot(p-n) = \frac{\cot p \cot n + 1}{\cot n - \cot p} = \frac{(d/t)(d/b)+1}{d/b - d/t} = \frac{d}{t-b} + \frac{bt}{d\,(t-b)}$$

Letting $u = d\,/\,(t-b)$ and $v = b\,t\,/\,d(t-b)$,

$\cot m = u + v \geq 2\sqrt{(u\,v)}$ (The arithmetic mean of two positive numbers, u and v, is never smaller than their geometric mean and the two means are equal only if the two numbers are equal)[1]

$$\cot m = 2\sqrt{\frac{d}{(t-b)} \cdot \frac{bt}{d(t-b)}} = \frac{2\sqrt{b\,t}}{t-b}$$

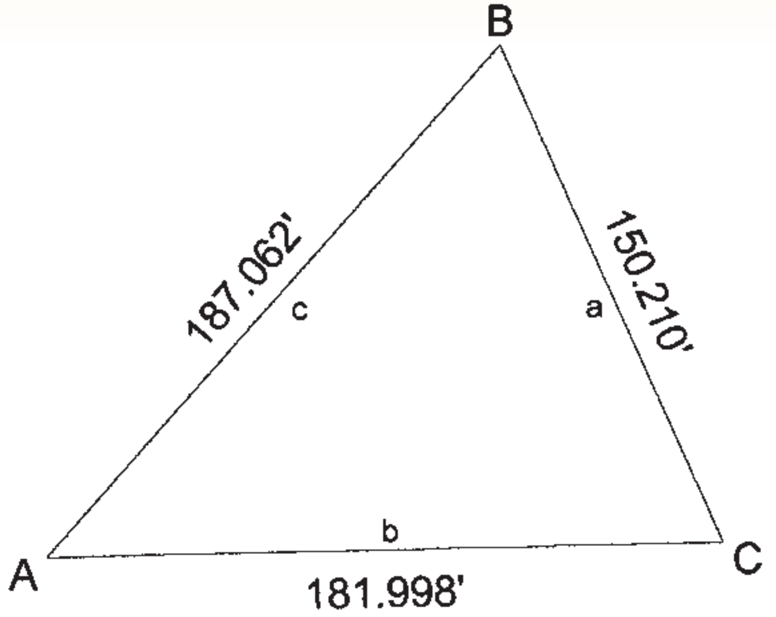
Equality holds only if $d\,/\,(t-b) = b\,t\,/\,d\,(t-b)$, or when $d = \sqrt{b\,t}$

Substituting numbers, b=10'–3.5' and t=18'–3.5', $d = \sqrt{6.5' \times 14.5'} = 9.7'$

[1] This follows from the fact that the square of a real number can never be negative: $0 \leq (\sqrt{u} - \sqrt{v})^2 = u - 2\sqrt{u\,v} + v$

$0 \leq \sqrt{u\,v} = \frac{1}{2}(u+v)$, with equality only if $\sqrt{u} = \sqrt{v} = 0$, or when $u = v$.

Solution to Problem 100



In any triangle, $\sin(\frac{1}{2}A) = \sqrt{(s-b)(s-c)/bc}$ and $\cos(\frac{1}{2}A) = \sqrt{s(s-a)/bc}$

where $s = (a + b + c) / 2$

Similarly, $\sin(\frac{1}{2}B) = \sqrt{(s-a)(s-c)/ac}$ and $\cos(\frac{1}{2}B) = \sqrt{s(s-b)/ac}$

and $\sin(\frac{1}{2}C) = \sqrt{(s-a)(s-b)/ab}$ and $\cos(\frac{1}{2}C) = \sqrt{s(s-c)/ab}$

For this triangle, $s = (150.210 + 181.998 + 187.062) / 2 = 259.635$

Since calculating the cosine is less work,

$\cos(\frac{1}{2}A) = \sqrt{(259.635)(259.635-150.210)/(181.998 \times 187.062)}$

$= \sqrt{259.635 \times 109.425/181.998 \times 187.062} = 0.913511043$

and A = 48°00'34.9"

$\cos(\frac{1}{2}B) = \sqrt{(259.635)(259.635-181.998)/(150.210 \times 187.062)}$

$= 0.846981221$, and B = 64°13'49.2"

$\cos(\frac{1}{2}C) = \sqrt{(259.635)(259.635-187.062)/(150.210 \times 181.998)}$

$= 0.830207067$, and C = 67°45'35.9"

The sum is indeed 180°00'00.0" and each angle may be checked by the sine formulae.