Solution to Problem 47

LET THE CENTRAL ANGLE = 2 ϕ

$2 R \phi = L = 101'$, with ϕ expressed in radians, so that $\phi = \dfrac{50.5}{R}$

$2R \sin\phi = \text{chord} = 100'$, with ϕ expressed in degrees, so that $\sin\phi = \dfrac{50}{R}$

$$\frac{\sin\phi}{\phi} = \frac{50 / R}{50.5 / R} = 0.990099009901$$

From trigonometry:

$$\sin\phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + \frac{\phi^9}{9!} - \ldots$$

$$0.990099009901 = 1 - \frac{\phi^2}{6} + \frac{\phi^4}{120}$$, *ignoring all powers above the fourth*

multiplying by 120 and rearranging, $\phi^4 - 20\phi^2 + 1.1881188118811 = 0$

and, by utilizing the quadratic equation

$$\phi = \frac{\sqrt{(20) \pm \sqrt{(20)^2 - (4)(1.1881188118811)}}}{\sqrt{2}}$$

$\phi = 4.465469$ and 0.244097 (For the purists: 0.2440966956936829825

or 255°51'09.1" and 13°59'08.6" 13°59'08.55764281396"

so that R = 206.885' (from the chord equation) 206.8852257769702271')

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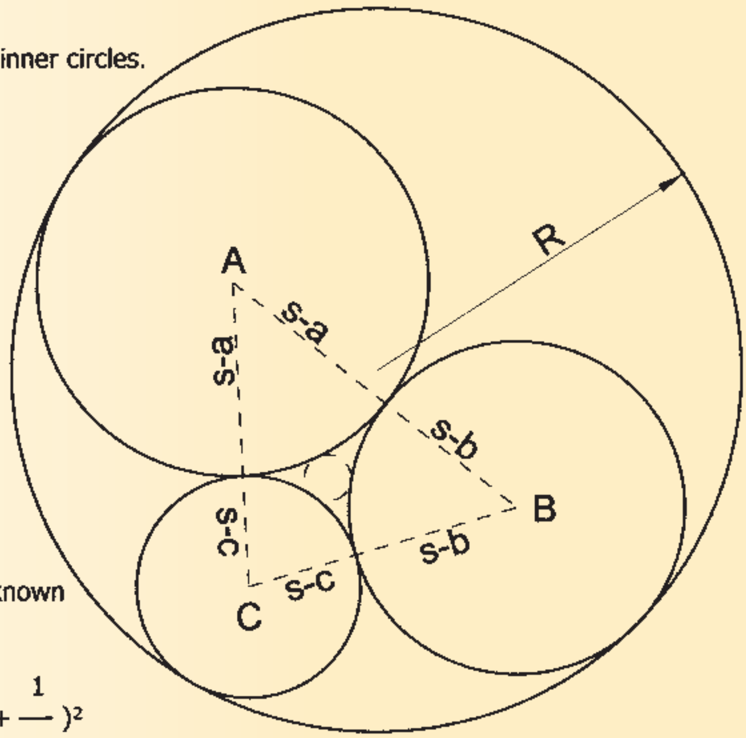
Solution to Problem 48



Construct triangle ABC by joining the radius points of the inner circles.

Let BC = a, AC = b, AB = c, and $s = \frac{1}{2}(a+b+c)$

$a = 2s - b - c$, $b = 2s - a - c$, $c = 2s - a - b$

$a = (s - b) + (s - c)$, $b = (s - a) + (s - c)$, $c = (s - a) + (s - b)$

from which $r_A = (s - a)$, $r_B = (s - b)$, $r_C = (s - c)$

From Descartes' Circle Theorem for four tangent circles:

$$2\left(\frac{1}{r_A^2} + \frac{1}{r_B^2} + \frac{1}{r_C^2} + \frac{1}{R^2}\right) = \left(\frac{1}{r_A} + \frac{1}{r_B} + \frac{1}{r_C} + \frac{1}{R}\right)^2$$

for this problem $r_A = 0.875''$, $r_B = 0.75''$, $r_C = 0.5''$, R is unknown

$$2\left(\frac{1}{0.875^2} + \frac{1}{0.75^2} + \frac{1}{0.5^2} + \frac{1}{R^2}\right) = \left(\frac{1}{0.875} + \frac{1}{0.75} + \frac{1}{0.50} + \frac{1}{R}\right)^2$$

$$2\left(7.083900227 + \frac{1}{R^2}\right) = \left(4.476190476 + \frac{1}{R}\right)^2$$

from which $R = -1.6300''$ and $0.1045''$

(The larger absolute value being the outer circle radius, the smaller absolute value being the inner circle radius, shown dashed in the drawing.)

(Note: If S denotes the center of the circle of radius R, it makes the perimeters of triangles ASB, ASC and CSB equal, hence the name "isoperimetric point")