Solution to Problem 32

To simplify the problem, let $R = AD = CD = 1$. Let angle $ADC$ be $\phi$.

Angle $ACD = angle CAD = \frac{\pi}{2} (\pi - \phi)$, with all angles expressed in radians.

The area of sector $A-C-E$ equals $\frac{1}{2} \ r^2 \ (\pi / 2 - \phi / 2)$.

The area of segment $A-F-C$ equals $\frac{1}{2} R^2 (\phi - \sin \phi)$, with $R = 1$.

The area of the sector plus the area of the segment must equal $\pi / 4$.

so, $\frac{r^2}{2} \left(\frac{\pi}{2} - \phi\right) + \frac{1}{2} \ (\phi - \sin \phi) = \frac{\pi}{4},$

or, $\pi r^2 - \phi r^2 + 2 \phi - 2 \sin \phi - \pi = 0$

since $2 \ R \sin (\phi / 2) = r$, with $R = 1$, $r^2 = 4 \sin^2 (\phi / 2) = 2 (1 - \cos \phi)$

substituting for $r^2$,

$\pi (2 - 2 \cos \phi) - \phi (2 - 2 \cos \phi) + 2 \phi - 2 \sin \phi - \pi = 0$

which reduces to $(\phi - \pi - \tan \phi - \frac{\pi}{2 \cos \phi}) = 0$, after collecting terms and dividing by $2 \cos \phi$.

Knowing that the value of $\phi$ has to be greater than zero and less than $\pi/2$ (that would make $ADB$ a straight line), plotting a few values such as $\phi = 0, 0.5, 1, \text{and} 1.5$ shows the value to be near 1.25 (There is an asymptotic line at $\phi = 1.5708 \pm$ where $\phi = 90^\circ$). Further refinement yields $\phi = 1.235896924$ and

$r = 1.158728414 \ R$

($\phi = 1.2359$ is within 1 second of arc for $\phi$ and within 0.0000025R for $r$)